

Neostability-properties of Fraïssé limits of 2-nilpotent groups of exponent $p > 2$

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Abstract

Let $L(n)$ be the language of group theory with n additional new constant symbols c_1, \dots, c_n . In $L(n)$ we consider the class $\mathbb{K}(n)$ of all finite groups G of exponent $p > 2$, where $G' \subseteq \langle c_1^G, \dots, c_n^G \rangle \subseteq Z(G)$ and c_1^G, \dots, c_n^G are linearly independent. Using amalgamation we show the existence of Fraïssé limits $D(n)$ of $\mathbb{K}(n)$. $D(1)$ is Felgner's extra special p -group. The elementary theories of the $D(n)$ are superstable of SU-rank 1. They have the independence property.

1 Introduction

We consider the variety $\mathbb{G}_{2,p}$ of nilpotent groups of class 2 of exponent $p > 2$ in the language L of group theory. To get the Amalgamation Property (AP) in [Bau] an additional predicate $P(G)$ for $G \in \mathbb{G}_{2,p}$ with $G' \subseteq P(G) \subseteq Z(G)$ is introduced. Let $\mathbb{G}_{2,p}^P$ be the category of this groups in the extended language L_P where the morphisms are embeddings. Using the class $\mathbb{K}_{2,p}^P$ of finite structures in $\mathbb{G}_{2,p}^P$ we get a Fraïssé limit D . If we build D by amalgamation then $P(a)$ says that a will become an element of the commutator subgroup D' of D in that process. In [Bau] it is shown that $\text{Th}(D)$ is not simple. Here we point out that D has the tree property of the second kind (TP₂). This is easily seen.

Let $L(n)$ be the language of group theory with n additional new constant symbols c_1, \dots, c_n . In $L(n)$ we consider the class $\mathbb{G}(n)$ of all groups $G \in \mathbb{G}_{2,p}$, where $G' \subseteq \langle c_1^G, \dots, c_n^G \rangle \subseteq Z(G)$ and c_1^G, \dots, c_n^G are linearly independent. We use linear independence, since we can consider an abelian group of exponent p as a

vector space over \mathbb{F}_p . $\langle X \rangle$ denotes the substructure generated by X . Hence $\langle c_1^G, \dots, c_n^G \rangle = \langle \emptyset \rangle$. $\mathbb{G}(n)$ is uniformly locally finite. Let $\mathbb{K}(n)$ be the class of finite structures in $\mathbb{G}(n)$. $\mathbb{K}(n)$ has the Hereditary Property (HP), the Joint Embedding Property (JEP) and the Amalgamation Property (AP). Hence the Fraïssé limit $D(n)$ of the class $\mathbb{K}(n)$ exists. Note that $D(1)$ is the extra special p -group considered by U. Felgner in [Fe]. In [MacSt] the corresponding bilinear alternating map is obtained as an ultraproduct of finite structures. It is a well-known example of a supersimple theory of SU-rank 1.

We show that the theories of all Fraïssé limits $D(n)$ are supersimple of SU-rank 1. To prove this we check the properties of non-forking that characterize simple theories [KP]. Before we show that each group G in $\mathbb{G}(n)$ where $G/Z(G)$ is infinite has the Independence Property especially all $D(n)$.

2 TP_2 of $\text{Th}(D)$

Proposition 2.1 *In $\text{Th}(D)$ the formulae $[x, y_1] = [y_2, y_3]$ has the tree property of the second kind.*

Proof. Since D is the Fraïssé limit of $\mathbb{K}_{2,p}^P$ there is an embedding of an infinite free group of $G_{2,p}$ in D . Assume that $\{b_\alpha : \alpha < \omega\} \cup \{c_{\alpha,i}, d_{\alpha,i} : \alpha < \omega, i < \omega\}$ are free generators of such an infinite free subgroup. We consider the array

$$\bar{a}_{\alpha,i} = \{(b_{\alpha,i}, c_{\alpha,i}, d_{\alpha,i}) : \alpha < \omega, i < \omega\}$$

where $b_{\alpha,i} = b_\alpha$ for all α and i :

$$\text{Then } D \models \neg \exists x ([x, b_\alpha] = [c_{\alpha,i}, d_{\alpha,i}] \wedge [x, b_\alpha] = [c_{\alpha,j}, d_{\alpha,j}])$$

for fixed α and $i \neq j$. Now let f be any map of ω into ω . Then the set

$$\{[x, b_\alpha] = [c_{\alpha,f(\alpha)}, d_{\alpha,f(\alpha)}] : \alpha < \omega\}$$

is consistent, since D is a Fraïssé limit. □

3 The amalgamation property for $\mathbb{K}(n)$

Let G be a group in $\mathbb{G}(n)$ with the elements c_1^G, \dots, c_n^G short c_1, \dots, c_n . Let $P(G)$ be the subgroup generated by c_1, \dots, c_n . In the language $L(n)$ $P(G)$ is the $L(n)$ -substructure generated by the empty set. By definition $G' \subseteq P(G) \subseteq Z(G)$ and the linear dimension $\text{ldim}(P(G))$ of $P(G)$ is n .

In [Bau] a functor F from $\mathbb{G}_{2,p}^P$ into the category \mathbb{B}^P of bilinear alternating maps (V, W, β) is defined where V and W are \mathbb{F}_p -vector spaces and β is a bilinear alternating map from V into W . Morphisms of \mathbb{B}^P from (V_1, W_1, β_1) to (V_2, W_2, β_2) consists of vector space embeddings $f : V_1$ into V_2 and $g : W_1$ into W_2 that commute with the bilinear maps β_i .

$$\begin{array}{ccc} V_1 & \times & V_1 \xrightarrow{\beta_1} W_1 \\ \downarrow f & & \downarrow f \quad \downarrow g, \\ V_2 & \times & V_2 \xrightarrow{\beta_2} W_2 \end{array}$$

F is defined in the following way: $F(G)$ is (V, W, β) where $V = G/P(G)$, $W = P(G)$ and β is induced by $[\ , \]$. If $f : G \rightarrow H$ then $F(f) = (\bar{f}, f \upharpoonright P)$ where $\bar{f} : G/P(G) \rightarrow H/P(H)$ is induced by f . F is a bijection on the level of objects up to isomorphisms.

If we consider the category $\mathbb{G}(n)$, then the morphisms $f : G \rightarrow H$ send c_i^G to c_i^H . Hence f induces an isomorphism of $P(G)$ onto $P(H)$. We call $\mathbb{B}(n)$ the corresponding category of bilinear alternating maps (V, P, β) where $P = \langle c_1, \dots, c_n \rangle$ is fixed. The morphisms have the form (g, id) . We define the functor F from $\mathbb{G}(n)$ to $\mathbb{B}(n)$ as above and obtain as in [Bau2]:

Lemma 3.1 i) F is a functor of $\mathbb{G}(n)$ onto $\mathbb{B}(n)$ that is a bijection for the objects of the categories up to isomorphisms.

ii) If $G_0 \in \mathbb{G}(n)$ and (g, id) is an embedding of $F(G_0)$ into some (V, P, β) , then there are some $G \in \mathbb{G}(n)$ and some embedding f of G_0 into G , such that $F(G) = (V, P, \beta)$ and $F(f) = (g, \text{id})$.

iii) In $\mathbb{G}(n)$ we consider $e_0 : G_0 \rightarrow G$, $e_1 : H_0 \rightarrow H$ where f_0 is an isomorphism of G_0 onto H_0 . In $\mathbb{B}(n)$ we assume that there is g such that

$$\begin{array}{ccc} F(G_0) & \xrightarrow{F(e_0)} & F(G) \\ \downarrow F(f_0) & & \downarrow (g, \text{id}) \\ F(H_0) & \xrightarrow{F(e_1)} & F(H) \end{array} .$$

Then there is an embedding f of G into H such that $F(f) = (g, \text{id})$ and

$$\begin{array}{ccc} G_0 & \xrightarrow{e_0} & G \\ \downarrow f_0 & & \downarrow f \\ H_0 & \xrightarrow{e_1} & H \end{array} .$$

Lemma 3.1 shows that AP for $\mathbb{B}(n)$ implies AP for $\mathbb{G}(n)$ as in [Bau]. To show AP for $\mathbb{B}(n)$ we cannot use the free amalgam as in [Bau].

Assume

$$\begin{aligned} (f_A, \text{id}) : & (V_B, P, \beta_B) \longrightarrow (V_A, P, \beta_A), \\ (f_C, \text{id}) : & (V_B, P, \beta_B) \longrightarrow (V_C, P, \beta_C). \end{aligned}$$

W.l.o.g. V_B is a common subspace of V_A and V_C . Let V_D be the vector space amalgam $V_C \bigoplus_{V_B} V_A$ with respect to f_A and f_C . We get the desired amalgam $\langle V_D, P, \beta_D \rangle$ if

$$\beta_D = \beta_A \text{ on } V_A \quad \text{and} \quad \beta_D = \beta_C \text{ on } V_C$$

and the rest is obtained in the following way: If X is a basis of V_A over V_B and Y is a basis of V_C over V_B then we can choose for each pair $x \in X$ and $y \in Y$ $\beta_0(x, y)$ in P as we want.

In our context AP implies JEP.

Theorem 3.2 $\mathbb{K}(n)$ has HP, JEP and AP. Hence the Fraïssé limit $D(n)$ exists. It is \aleph_0 -categorical. $\text{Th}(D(n))$ has the elimination of quantifiers.

The theorem uses the known theory. See [Ho]. Uniform local finiteness and finite signature for $\mathbb{K}(n)$ imply \aleph_0 -categoricity and elimination of quantifiers. $\text{Th}(D(n))$ can be axiomatized by the following sentences: Let M be a model of $\text{Th}(D(n))$.

$\Sigma 1$) M is a nilpotent group of class 2 with exponent p .

$\Sigma 2$) $M' = Z(M) = \langle c_1, \dots, c_n \rangle$ is of linear dimension n .

$\Sigma 3$) For $B \subseteq A$ in $\mathbb{K}(n)$ it holds: If $B' \subseteq M$ and $B' \cong B$, then this embedding of B into M can be extended to A .

In the case $n = 1$ these axioms imply that M is infinite and $M' = Z(M)$ is cyclic. By U. Felgner [Fe] $D(1)$ is the extra special p -group, since his axiomatization is $\Sigma 1$) $M' = Z(M)$ is cyclic and infiniteness.

Question Is there an easier axiomatization of $\text{Th}(D(n))$ for $n \geq 2$?

4 Independence property in $\mathbb{G}(n)$

Assume $M \models \text{Th}(D(1))$ and M is countable. We write c instead of c_1 . By [Fe] M is a central product over $\langle c \rangle$:

$$M = \bigodot_{\substack{\langle c \rangle \\ i < \omega}} \langle c, a_i b_i \rangle$$

where c is a generator of the cyclic subgroup $M' = Z(M)$ and $[b_i, a_i] = c$.

By the elimination of quantifiers of $\text{Th}(D(1))$ $a_0 \hat{=} b_0, a_1 \hat{=} b_1, \dots, a_n \hat{=} b_n, \dots$ is an indiscernible sequence in M . Then $a_1 \hat{=} b_1, a_2 \hat{=} (b_2 \circ b_0), a_3 \hat{=} b_3, a_4 \hat{=} (b_4 \circ b_0), \dots$ and $b_1, b_2 \circ b_0, b_3, b_4 \circ b_0, \dots$ are indiscernible sequences in M . We have $M \models [b_{2i+1}, a_0] = 1$ for $i < \omega$ and $M \models [b_{2i} \circ b_0, a_0] = c$ for $i \leq i < \omega$.

We have shown (see [A]):

Lemma 4.1 *The formula $[y, x] = 1$ has the independence property in $\text{Th}(D(1))$.*

Let G be in $\mathbb{G}(n)$ with $G/Z(G)$ is infinite. For $a \in G \setminus Z(G)$ choose a maximal linearly independent subset $\{e_1, \dots, e_m\} = X_a$ of $P(G)$ such that for every $1 \leq i \leq m \leq n$ there is some $b_i \in G$ with $[a, b_i] = e_i$. Let E_a be $\{b_1, \dots, b_m\}$. If $[a, b] = t \neq 1 \in P(G)$, then $t = \sum_{1 \leq i \leq m} e_i^{r_i}$ and $[a, b \cdot b_1^{p-r_1} \cdot \dots \cdot b_m^{p-r_m}] = 1$. Hence every element $a \in G$ has a centralizer $C(a)$ of index $\leq n$ and $G = \langle a, E_a \rangle \circ C(a)$.

Now we start again with $d_0 \in G \setminus Z(G)$. Then $X_{d_0} \neq \emptyset$ and $E_0 = E_{d_0} \neq \emptyset$ and we choose $e_0 \in E_0$ with $[d_0, e_0] \neq 1$. Since $C(\langle d_0, E_0 \rangle)$ has finite index in G there is some

$$d_1 \in C(\langle d_0, E_0 \rangle) \quad \text{with } d_1 \notin Z(G).$$

We get $E_1 = E_{d_1} \neq \emptyset$ and choose $b_1 \in E_1$. We can repeat this argument and get

$$d_2 \in C(\langle d_0, e_0, E_0, E_1 \rangle), \quad d_2 \notin Z(G).$$

Finally we have $d_0, e_0, d_1, e_1, \dots$ with $[d_i, e_i] \neq 1$ and $[d_i, d_j] = 1$, $[d_i, e_j] = 1$ and $[e_i, e_j] = 1$ for $i \neq j$. We can select a subsequence with $[d_i, e_i] = c \neq 1$ for some $c \in P(G)$. Assume w.l.o.g. $[d_i, e_i] = c$ for all $i < \omega$. We have shown that $D(1)$ is a subgroup of G . Since the independence property of $D(1)$ is given by a quantifier formula we get

Theorem 4.2 *For every $G \in \mathbb{G}(n)$ with $G/Z(G)$ infinite we have:*

- i) *There is an embedding of $D(1)$ in G .*
- ii) *G has the independence property.*

Corollary 4.3 *The Fraïssé limits $D(n)$ of $\mathbb{K}(n)$ have the independence property.*

5 Superstability of $Th(D(n))$

Let $\mathbb{C}(n)$ be a monster model of $Th(D(n))$. We define

$$A \underset{B}{\downarrow}^0 C, \text{ if } \langle A \rangle \cap \langle C \rangle = \langle B \rangle.$$

Note that all substructures as $\langle A \rangle$ contain $P(\mathbb{C}(n))$.

We have to check that $\underset{B}{\downarrow}^0$ fulfils the conditions of B. Kim and A. Pillay [KP] that characterize Non-forking. Working in the vector space $\mathbb{C}(n)/P(\mathbb{C}(n))$ Monotonicity, Transitivity, Symmetry, Finite Character, and Local Character are easily shown.

Existence: $\bar{a}, B \subseteq A$ are considered in $\mathbb{C}(n)$. Then there is some \bar{d} in $\mathbb{C}(n)$ with $\text{tp}(\bar{a}/B) = \text{tp}(\bar{d}/B)$ and $\bar{d} \underset{B}{\downarrow}^0 A$.

W.l.o.g. B and A are $L(n)$ -substructures. Since $P \subseteq B$ we can assume that \bar{a} is linearly independent over B . Choose X_B and X_A such that the images of X_B and $X_B X_A$ are vector space bases of B/P and A/P , respectively. Let $\beta((X_B X_A)^2)$ be the set of all $\beta(b_1, b_2) = [b_1, b_2]$ where $b_1, b_2 \in X_B X_A$. Then A is uniquely determined by $X_B X_A$ and $\beta((X_B X_A)^2)$.

Now we define an extension G of A . Let \bar{e} linearly independent over A . $\bar{e} X_B X_A$ is linearly independent over P . $\beta((\bar{e} X_B X_A)^2)$ is chosen as any extension of $\beta((X_B X_A)^2)$ and $\beta((\bar{e} X_B)^2)$, where the last set is obtained from $\beta((\bar{a} X_B)^2)$ by replacing a_i in \bar{a} by e_i in \bar{e} . $G = \langle \bar{e} A \rangle$ is a structure in $\mathbb{K}(n)$. By the axioms $\Sigma 3$ of $Th(D(n))$ there is an embedding of \bar{e} onto \bar{d} over A in $\mathbb{C}(n)$. By quantifier elimination $\text{tp}(\bar{d}/B) = \text{tp}(\bar{a}/B)$. Furthermore $\bar{d} \underset{B}{\downarrow}^0 A$ by construction.

Finally we have to show:

Independence over Models

Let $M \preceq \mathbb{C}(n)$, $\text{tp}(\bar{a}^0/M) = \text{tp}(\bar{a}^1/M)$

$$\bar{b}^0 \downarrow_M^0 \bar{b}^1, \quad \bar{a}^0 \downarrow_M^0 \bar{b}^0, \quad \bar{a}^1 \downarrow_M^0 \bar{b}^1.$$

Then there is some \bar{e} with

$$\text{tp}(\bar{e}/M\bar{b}^0) = \text{tp}(\bar{a}_0/M\bar{b}^0), \quad \text{tp}(\bar{e}/M\bar{b}^1) = \text{tp}(\bar{a}_1/M\bar{b}^1),$$

and

$$\bar{e} \downarrow_M^0 \bar{b}^0 \bar{b}^1.$$

Let X_M be a set in M such that its image is a vector space basis of M/P . By assumption we can assume that w.l.o.g. $\bar{b}^0 \bar{b}^1$ is linearly independent over M modulo P . We choose \bar{d} linearly independent over $\bar{b}^0 \bar{b}^1 X_M$ modulo P . Now we extend $\langle \bar{b}^0, \bar{b}^1, X_M \rangle$ to a group G in $\mathbb{G}(n)$ defined on $\bar{d} \bar{b}^0 \bar{b}^1 X_M$. We extend $\beta((\bar{b}^0 \bar{b}^1 X_M)^2)$ to $\beta((\bar{d} \bar{b}^0 \bar{b}^1 X_M)^2)$ by the following:

$$\begin{aligned} \beta(d_i, m) & \quad \text{for } d_i \in \bar{d} \text{ and } m \in X_M \text{ is given by } \beta(a_i^0, m) = \beta(a_i^1, m), \\ \beta(d_i, b_j^0) & \quad \text{is given by } \beta(a_i^0, b_j^0), \text{ and} \\ \beta(d_i, b_j^1) & \quad \text{is given by } \beta(a_i^1, b_j^1). \end{aligned}$$

Now we find an image \bar{e} of \bar{d} in $\mathbb{C}(n)$ over $\langle \bar{b}^0, \bar{b}^1, M \rangle$ by axioms $\Sigma 3$) that defines an embedding. By elimination of quantifiers and the construction \bar{e} has the desired properties.

Theorem 5.1 \downarrow^0 is non-forking for $D(n)$. $D(n)$ is supersimple of SU-rank 1. It is not stable.

Proof. As shown above \downarrow^0 is non-forking and $D(n)$ is simple. Any type $\text{tp}(\bar{a}/A)$ does not fork on a finite subset of A . By the description of non-forking we have SU-rank 1. In Chapter 4 it is shown that $D(n)$ has the independence property.

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